Problem 17. Show that for the spectral radius of an operator $T \in B(\mathcal{H})$ in a Hilbert space $\mathcal{H}$ it holds that

$$r(T) = \inf \left\| VTV^{-1} \right\|$$

where the infimum runs over all invertible operators $V : H \to H$.
Hint: It suffices to consider the case $r(T) < 1$. First define and examine the new scalar product

$$\langle x, y \rangle_0 = \sum_{n=0}^{\infty} \langle T^n x, T^n y \rangle.$$

Problem 18. Let $T : X \to Y$ be a bounded operator between Banach spaces X, Y and let $||| \cdot |||$ denote the quotient norm of the quotient Banach space $X / \ker(T)$. If for each $y = Tx \in \operatorname{ran}(T)$, we let

$$||y||_T = ||y|| + |||[x]|||,$$

then:
(a) The function $|| \cdot ||_T$ defines a norm on $\operatorname{ran}(T)$,
(b) $(\operatorname{ran}(T), || \cdot ||_T)$ is a Banach space,
(c) The operator $T : (X, || \cdot ||) \to (\operatorname{ran}(T), || \cdot ||_T)$ is continuous.

Problem 19. Let $T : X \to Y$ be a bounded operator between Banach spaces X, Y. If the quotient space $Y / \operatorname{ran}(T)$ is finite dimensional, then the $\operatorname{ran}(T)$ is closed.

Problem 20. Let $T : X \to Y$ be a bounded operator between Banach spaces X, Y such that $\dim \ker(T) < \infty$ and $\operatorname{codim} \operatorname{ran}(T) < \infty$. Show that T is a Fredholm operator, i.e. that the condition $\operatorname{ran}(T)$ is closed in the definition of a Fredholm operator is automatically satisfied.

Problem 21. Let H be a Hilbert space and $K(H) \subseteq B(H)$ be the closed ideal of compact operators on H so then $C(H) = B(H)/K(H)$ is a Banach algebra. Hence, $[T_1] = [T_2]$ if and only if $T_1 + T_2 + K$ for some compact perturbation K. Show that the following are equivalent:
(i) $[T]$ is invertible in $C(H)$,
(ii) There exists a $S \in B(H)$ such that $I - TS \in K(H)$ and $I - ST \in K(H)$,
(iii) T is Fredholm.